称号及び氏名　　博士(理学)　　曲　明珠

学位授与の日付　2024 年 9 月 23 日

論　　文　　名　Stability Analysis of Delay Differential Equations with Two Delays
（2つの時間遅れをもつ微分方程式の安定性解析）

論文審査委員　　主査　　松永　秀章
副査　　壁谷　喜継
副査　　今野　良彦
副査　　谷川　智幸

論文要旨

Stability Analysis of Delay Differential Equations with Two Delays
（２つの時間遅れをもつ微分方程式の安定性解析）

Mingzhu Qu
Department of Mathematical Sciences, Osaka Prefecture University

Differential equations are used as mathematical models to describe dynamical systems in the real world. Usually in the research process, if ordinary differential equations are used to describe a dynamical system, it is first necessary to assume that the state of the system at each point in time is only affected by the independent variables of the current system. But in fact, many natural phenomena show that the development mechanism of a system is not only related to the current state of the system, but also closely related to the past state of the system. For example, in the study of ecosystem models, in order to accurately reflect the form of natural systems, the delay term is an influential factor that cannot be ignored, because the reproduction, migration and other behaviors of animals and plants will have delays. In engineering systems controlled by feedback items, if the delay caused by the feedback process is too long, the system will produce large oscillations and become unstable. In the field of economics, the law of value operates due to delay between the production process of producers and the purchasing behavior of consumers. Generally speaking, the delay differential equations have more complex dynamical properties and behaviors than the corresponding ordinary differential equations, which means that the change of the equilibrium points are more flexible and difficult to predict. This is because the delay terms that affect the equation can also change the stability and topological structure of the equilibrium of the equation, which may produce Hopf bifurcation, fixed point bifurcation, homoclinic orbits, heteroclinic orbits, limit cycles, chaos, and other phenomena. The stability analysis of delay differential equations is the basis for clarifying these phenomena, and therefore, it is a very valuable research topic. In addition, the behavior of the solution is more complicated for differential equations with multiple delays than for differential equations with a single delay.

The purpose of this thesis is to study the stability analysis of delay differential equations with two delays. The thesis consists of three chapters: In Chapter 1 we investigate the dynamical analysis of a Beddington-DeAngelis commensalism system with two delays and in Chapters 2 and 3 we discuss the exact stability criteria for scalar linear differential equations with discrete and distributed delays.

Commensalism is a class of relationship between two organisms where one organism benefits but the other is neutral (there is no harm or benefit). Amensalism is a system between two species in which organisms of one species cause pain or death to organisms of other species without any benefit to themselves. Many studies have investigated the dynamical properties of commensalism and amensalism models. A mathematical model for representing a commensalism or amensalism system was introduced in 2003 by Sun and Wei:

$$\begin{cases} x' = r_1 x \left(\dfrac{k_1 - x + \alpha y}{k_1} \right), \\ y' = r_2 y \left(\dfrac{k_2 - y}{k_2} \right), \end{cases}$$

where $' = d/dt$, x and y, correspond to the populations of two species at time t, respectively; r_1, r_2, k_1, and k_2 are positive constants. The paradigm is referred to as commensalism if α is positive, and amensalism if α is negative.

In Chapter 1, we establish a two-species commensalism model, which incorporates the Beddington–DeAngelis functional response with two delays

$$\begin{cases} x'(t) = x(t) \left(a_1 - b_1 x(t - \tau_1) - c_1 x(t - \tau_2) \right), \\ y'(t) = y(t) \left(a_2 - b_2 y(t) + \dfrac{c_2 x(t)}{m x(t) + n y(t) + 1} \right), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ denote the densities of two species at time t, respectively; a_1 and a_2 indicate the intrinsic growth of the two species, b_1 and b_2 represent the density restriction coefficients of the two populations, respectively. The functional response of the Beddington–DeAngelis type, that is, $cx(t)/\left(mx(t) + ny(t) + 1\right)$ describes the beneficial influence of the first species on the second species. The delay τ_1 indicates the maturation time of the population. Here, we also consider the mobility of population $x(t)$ as $c_1 x(t)$; hence, the term $c_1 x(t - \tau_2) x(t)$ not only considers the migration that depends on the population density but also considers the delay of the impact of population migration. The parameters a_1, a_2, b_1, b_2, c_1, c_2, m, and n are all positive constants. The initial conditions of (1) are given by

$$\begin{aligned} &x(\theta) = \phi_1(\theta) \geq 0, \;\; y(\theta) = \phi_2(\theta) \geq 0, \;\; \theta \in [-r, 0], \;\; r = \max\{\tau_1, \tau_2\}, \\ &\phi_1(0) > 0, \;\; \phi_2(0) > 0, \end{aligned}$$

where $(\phi_1(\theta), \phi_2(\theta)) \in C\left([-r, 0], \mathbb{R}^2_+\right)$, $\mathbb{R}^2_+ = \{(x, y) \mid x \geq 0, y \geq 0\}$.

In contrast with existing literature on commensalism systems, the system considered in the present study has two delays in one species. We investigate the local stability of the positive equilibrium and the presence of Hopf bifurcation. Furthermore, we construct the Lyapunov functional to show the system has uniform persistence. Finally, we provide numerical simulations to demonstrate the reliability of the derived results.

In Chapters 2 and 3, we study the asymptotic stability of a scalar linear differential equation with discrete and distributed delays

$$x'(t) = -ax(t) - bx(t-\tau) - c\int_{t-h}^{t} x(s)ds, \quad t \geq 0. \tag{2}$$

Here, a, b, and c are real numbers, τ and h are positive constants. Because (2) is linear and autonomous, the asymptotic stability of (2) is completely characterized by the roots of the associated characteristic equation

$$\lambda + a + be^{-\lambda\tau} + c\int_{-h}^{0} e^{\lambda s}ds = 0. \tag{3}$$

More precicely, the zero solution of (2) is asymptotically stable if and only if all the roots of (3) have negative real parts.

In Chapter 2, we first discuss the asymptotic stability of (2) with the special discrete and distributed delay, that is, $h = \tau$; equation (2) becomes

$$x'(t) = -ax(t) - bx(t-\tau) - c\int_{t-\tau}^{t} x(s)ds, \quad t \geq 0. \tag{4}$$

In 2004, Sakata and Hara provided the stability result for (4) with $a = 0$ or $b = 0$, and Huang and Vandewalle derived the stability result for (4) with fixed τ. In 2006, Funakubo et al. presented the delay-dependent stability criterion for (4) with $b = 0$ and fixed a and c. To our best knowledge, no delay-dependent stability criterion for (4) with fixed $b \neq 0$ and $c \neq 0$ has been obtained. We establish necessary and sufficient conditions guaranteeing the asymptotic stability of the zero solution composed of explicit delay-dependent criteria. Let $\omega_0 = \sqrt{b^2 + 2c - a^2}$, and let τ^*, τ_0, and σ_0 be the critical values of τ defined as

$$\tau^* = -\frac{a+b}{c}, \quad \tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{(b-a)^2 - \omega_0^2}{(b-a)^2 + \omega_0^2}\right), \quad \sigma_0 = \frac{1}{\omega_0}\left(2\pi - \arccos\left(\frac{(b-a)^2 - \omega_0^2}{(b-a)^2 + \omega_0^2}\right)\right).$$

Theorem. *The zero solution of* (4) *is asymptotically stable if and only if* $a + b + c\tau > 0$ *and any one of the following nine conditions holds:*

(i) $a > 0, \quad b \geq a, \quad b^2 + 2c - a^2 > 0,$ *and* $0 < \tau < \tau_0,$

(ii) $a > 0, \quad b > -a, \quad b^2 + 2c - a^2 \leq 0, \quad c < 0,$ *and* $0 < \tau < \tau^*,$

(iii) $a > 0, \quad b^2 + 2c - a^2 \leq 0, \quad c \geq 0,$ *and* τ *is arbitrary,*

(iv) $a > 0, \quad |b| < a, \quad b^2 + 2c - a^2 > 0,$ *and* $0 < \tau < \sigma_0,$

(v) $a > 0, \quad b \leq -a, \quad c > 0,$ *and* $\tau^* < \tau < \sigma_0,$

(vi) $a \leq 0, \quad b > -a, \quad b^2 + 2c - a^2 > 0,$ *and* $0 < \tau < \tau_0,$

(vii) $a \leq 0, \quad b > -a, \quad b^2 + 2c - a^2 \leq 0, \quad \textit{and} \quad 0 < \tau < \tau^*,$

(viii) $a \leq 0, \quad |b| \leq -a, \quad b^2 + 2c - a^2 > 0, \quad \textit{and} \quad \tau^* < \tau < \tau_0,$

(ix) $a \leq 0, \quad b < a, \quad c > 0, \quad \textit{and} \quad \tau^* < \tau < \sigma_0.$

In Chapter 3, we further investigate the asymptotic stability of (2) with the different discrete and distributed delay, that is, $h = 2\tau$; equation (2) becomes

$$x'(t) = -ax(t) - bx(t - \tau) - c \int_{t-2\tau}^{t} x(s)ds, \quad t \geq 0. \tag{5}$$

In 2004, Sakata and Hara obtained the stability region for (5) with $a = 0$. Compared with previous studies, no delay-dependent stability criterion for (5) with fixed $b \neq 0$ and $c \neq 0$ has been obtained. By using the root analysis of the corresponding characteristic equation, we derive necessary and sufficient conditions consisting of explicit delay-dependent criteria to ensure the asymptotic stability of the zero solution.

Original Papers

[1] Mingzhu Qu, Dynamical analysis of a Beddington–DeAngelis commensalism system with two time delays, *J. Appl. Math. Comput.* 69(6), 4111–4134 (2023)
https://doi.org/10.1007/s12190-023-01913-4

[2] Mingzhu Qu, H. Matsunaga, Exact stability criteria for linear differential equations with discrete and distributed delays, *J. Math. Anal. Appl.* 539(2), Paper No.128663, pp.15 (2024)
https://doi.org/10.1016/j.jmaa.2024.128663

[3] Mingzhu Qu, Delay–dependent stability for a linear differential equation with discrete and distributed delays, submitted for publication

学位論文審査結果の要旨

学位論文提出者氏名：曲　明珠

学位論文題目： Stability Analysis of Delay Differential Equations with Two Delays
（２つの時間遅れをもつ微分方程式の安定性解析）

時間遅れをもつ微分方程式とは、現象の変化が現在の状態だけでなく、過去の状態にも依存する現象を定式化した方程式である。時間遅れのパラメータは例えば、生物モデルにおける個体の成長期間や制御モデルにおける伝達遅延時間を表す。時間遅れをもつ非線形方程式の解の漸近的性質（漸近安定性、周期解の存在、分岐現象など）の解析は、平衡点まわりの線形化方程式の安定性解析から始まる。したがって、線形微分方程式の零解の漸近安定性に関する研究は、基本的かつ重要な研究テーマの１つである。時間遅れをもつ微分方程式の漸近安定性に関する結果は、２つのタイプがある。１つは時間遅れのパラメータを固定したときの係数パラメータ空間における安定領域を与える結果である。もう１つは係数パラメータを固定したときの時間遅れに依存した漸近安定条件を与える結果である。特に、理論的および応用的な側面から、時間遅れに依存した漸近安定条件の導出が期待されている。

申請者はまず１章において、Beddington-DeAngelis 型片利共生システムに個体の成長期間と集団移動を表す異なる時間遅れを導入した数理モデルを考案し、２つの時間遅れが片利共生システムに与える影響を解析した。具体的には、時間遅れなし、１つの時間遅れ、および２つの時間遅れをもつ各場合の片利共生システムの平衡点が局所漸近安定であるための十分条件とホップ分岐の存在条件を導出した。加えて、片利共生システムの２つの種がパーシステント（長期間共存可能）であることを証明した。

次に、申請者は２章と３章において、時間遅れのない項 $ax(t)$、離散的な時間遅れをもつ項 $bx(t-\tau)$、連続的な時間遅れをもつ項 $c\int_{t-h}^{t} x(s)\,ds$ の３つの項からなるスカラー線形微分方程式の漸近安定性について考察した。２章では、$h=\tau$ のとき、零解が漸近安定であるための具体的な必要十分条件を導出した。零解の漸近安定性の証明は、付随する特性方程式のすべての根が負の実部をもつための必要十分条件を求めることに帰着される。既存の特性根解析をさらに精密に行うことで、複素平面の虚軸上と右半平面上における特性根の分布を完全に調べることができた。これにより、１つの時間遅れをもつ線形微分方程式の漸近安定性に関する従来の結果を、２種類の時間遅れをもつ線形微分方程式へ一般化することに成功した。３章では、$h=2\tau$ のときの零解の漸近安定性について考察した。申請者が得た研究成果はいずれも時間遅れに依存した漸近安定条件であり、２つの時間遅れをもつ数理モデルへの応用が期待される。

以上より、本委員会は本論文を学位論文として十分な内容を有しているものと判断した。

主査　松永　秀章
壁谷　喜継
今野　良彦
谷川　智幸